Estimating Intrinsic Source Size

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1. Introduction

Given several observations of a source, it is useful to combine them to derive an optimal estimate of the source's intrinsic angular size. Each observed source image is distorted by a number of effects including the telescope's point-spread function (PSF), detector pixelation, the presence of bad pixels and errors in aspect reconstruction. PSF distortion is often the most important of these effects. Unfortunately, PSF deconvolution is rarely practical because neither the source spectrum nor the PSF is accurately known and because the available data typically have a low signal-to-noise ratio. This memo describes an alternate approach that relies on an approximate measure of the intrinsic source angular size.

2. Approximate Correction for PSF Distortion

Using a telescope with PSF, $p(x, y)$, to observe a source, $s(x, y)$, one obtains a source image, $c(x, y)$, which is the convolution of the source and the PSF,

$$c(x, y) = \iint s(x', y')\, p(x - x', y - y')\, \mathrm{d}x'\mathrm{d}y'. \tag{1}$$

The goal is then to remove the effects of the PSF to better constrain the intrinsic shape of the source. To approach this problem analytically, consider the idealized case of a monochromatic source in which both the source and the PSF are elliptical Gaussians.

An elliptical Gaussian centered on the origin, with semi-axes σ_1 and σ_2, has the form

$$g(x, y; \sigma_1, \sigma_2, \phi) = \frac{g_0}{\sigma_1\sigma_2} \exp\left[-\pi(\mathcal{A}\mathbf{x})^2\right],$$

where

$$\mathcal{A} = \mathcal{U}\mathcal{R}_\phi = \begin{pmatrix} 1/\sigma_1 & \\ & 1/\sigma_2 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \tag{2}$$

and ϕ is the clockwise angle between the positive x-axis and the ellipse major axis.

For an elliptical Gaussian source, $s(x, y; a_1, a_2, \phi)$, and an elliptical Gaussian PSF, $p(x, y; b_1, b_2, \psi)$, one can show [see equation (A13)] that the PSF-convolved source, $c(x, y)$, is also an elliptical Gaussian,

$$c(x, y; \sigma_1, \sigma_2, \delta) = \frac{s_0 p_0}{\sigma_1\sigma_2} \exp\left[-\pi(\mathcal{T}\mathbf{x})^2\right], \tag{3}$$

where

$$\mathcal{T} = \begin{pmatrix} 1/\sigma_1 & \\ & 1/\sigma_2 \end{pmatrix} \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix}, \tag{4}$$

and where σ_1, σ_2 and δ are nonlinear functions of a_1, a_2, b_1, b_2, and $\alpha \equiv \phi - \psi$. In general, $\delta \neq \phi$; the PSF-convolved ellipse need not have the same orientation as the intrinsic source ellipse.

In principle, one can determine the parameters of the intrinsic source ellipse, $\{a_1, a_2, \phi\}$, by solving a nonlinear system of equations involving the PSF parameters, $\{b_1, b_2, \psi\}$, and the measured source parameters, $\{\sigma_1, \sigma_2, \delta\}$. However, because these equations are based on a crude approximation and because the input parameters are often uncertain, such an elaborate calculation seems unjustified.

A much simpler and more robust approach makes use of the identity

$$\sigma_1^2 + \sigma_2^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2, \tag{5}$$

which applies to the convolution of two elliptical Gaussians having arbitrary relative sizes and position angles [see equation (A15)]. Using this identity, one can define a *root-sum-square* intrinsic source size,

$$a_{\rm rss} \equiv \sqrt{a_1^2 + a_2^2} = \sqrt{\max\{0,\ (\sigma_1^2 + \sigma_2^2) - (b_1^2 + b_2^2)\}}, \tag{6}$$

that depends only on the sizes of the relevant ellipses and is independent of their orientations. This expression is analogous to the well-known result for convolution of 1D Gaussians and for convolution of circular Gaussians in 2D.

Using equation (6), one can derive an analytic expression for the uncertainty in $a_{\rm rss}$ in terms of the measurement errors associated with σ_i and b_i. Because σ_i and b_i are non-negative, evaluating the right-hand side of equation (6) using the corresponding mean values should give a reasonable estimate of the mean value of $a_{\rm rss}$. A Taylor series expansion of the right-hand side of equation (6) evaluated at the mean parameter values therefore yields the uncertainty

$$\Delta a_{\rm rss} = \frac{1}{a}\left[\sigma_1^2(\Delta\sigma_1)^2 + \sigma_2^2(\Delta\sigma_2)^2 + b_1^2(\Delta b_1)^2 + b_2^2(\Delta b_2)^2\right]^{1/2} \tag{7}$$

where $(\Delta X)^2$ represents the variance in X and where

$$a \equiv \begin{cases} a_{\rm rss}, & a_{\rm rss} > 0, \\ \sqrt{b_1^2 + b_2^2}, & a_{\rm rss} = 0. \end{cases} \tag{8}$$

Given a compatible set of measurements of intrinsic source size, $\{a_{{\rm rss},i} \pm \Delta a_{{\rm rss},i}\}$, the minimum variance estimator of the mean source size (Davis 2007) is the variance-weighted mean,

$$\overline{a_{\rm rss}} = {\rm Var}[\overline{a_{\rm rss}}] \sum_i {\rm Var}[a_{{\rm rss},i}]^{-1} a_{{\rm rss},i}, \tag{9}$$

where ${\rm Var}[a_{{\rm rss},i}] = (\Delta a_{{\rm rss},i})^2$. The variance in $\overline{a_{\rm rss}}$ is

$${\rm Var}[\overline{a_{\rm rss}}] = \left[\sum_i {\rm Var}[a_{{\rm rss},i}]^{-1}\right]^{-1}. \tag{10}$$

Equations (6) and (7) yield an estimate of the intrinsic source size, $a_{\rm rss} \pm \Delta a_{\rm rss}$, projected onto the tangent plane of an observation. For this measurement, the orientation of the source and PSF ellipses is irrelevant. As long as the angular size of the source is small enough that the small angle approximation is valid, the tangent plane estimate of the source size should differ by a negligible amount from the corresponding arc length on the celestial sphere (Davis 2007). For this reason, there is no need to transform the individual source size measurements, $a_{{\rm rss},i}$, to a common coordinate system before computing the variance-weighted mean in equation (9).

In my opinion, for reasons outlined above, derivation of the full set PSF-corrected ellipse parameters is unjustified in the context of level 3 archive processing. But, for completeness, I will describe one method of doing so.

To derive the parameters of the mean PSF-corrected source ellipse, $\{\overline{x_0}, \overline{y_0}, \overline{a_1}, \overline{a_2}, \overline{\phi}\}$, the parameters of the individual PSF-corrected ellipses must first be specified in a common coordinate system. A common coordinate system is important because the angle between an ellipse major axis and the local line of declination through the center of the ellipse is a strong function of the (α, δ) coordinates of the ellipse, particularly when the ellipse is located near the celestial poles. To determine the full set of PSF-corrected ellipse parameters for a given source observation, solve the nonlinear system of equations (A11), (A12) and (A14) for $\{a_1,\, a_2,\, \phi\}$, using the available values of $\{\sigma_1,\, \sigma_2,\, \delta,\, b_1,\, b_2,\, \psi\}$. The PSF-corrected ellipse parameters can then be transformed to a common coordinate system such as the (x, y) coordinates in the plane tangent at the mean source position, or the (α, δ) coordinates on the celestial sphere (for details, see Davis (2007)).

Next, express the individual PSF-corrected ellipses as polynomials of the form

$$1 = [\mathcal{A}(\mathbf{x} - \mathbf{x_0})]^2 = c_0 x^2 + c_1 y^2 + c_2 xy + c_3 x + c_4 y + c_5, \tag{11}$$

where $\mathbf{x_0}$ is the ellipse center and $\mathcal{A}$ is defined in equation (2). The polynomial coefficients of the mean ellipse, $\overline{c_k}$, may now be obtained from a weighted sum

$$\overline{c_k} = \sum_{i=1}^{N} w_i c_{k,i} \tag{12}$$

where the $c_{k,i}$ are the polynomial coefficients of the individual ellipses. The parameters of the mean PSF-corrected ellipse, $\{\overline{x_0}, \overline{y_0}, \overline{a_1}, \overline{a_2}, \overline{\phi}\}$, may be derived from the mean polynomial coefficients, $\overline{c_k}$, by solving a nonlinear system of 5 equations in 5 unknowns.

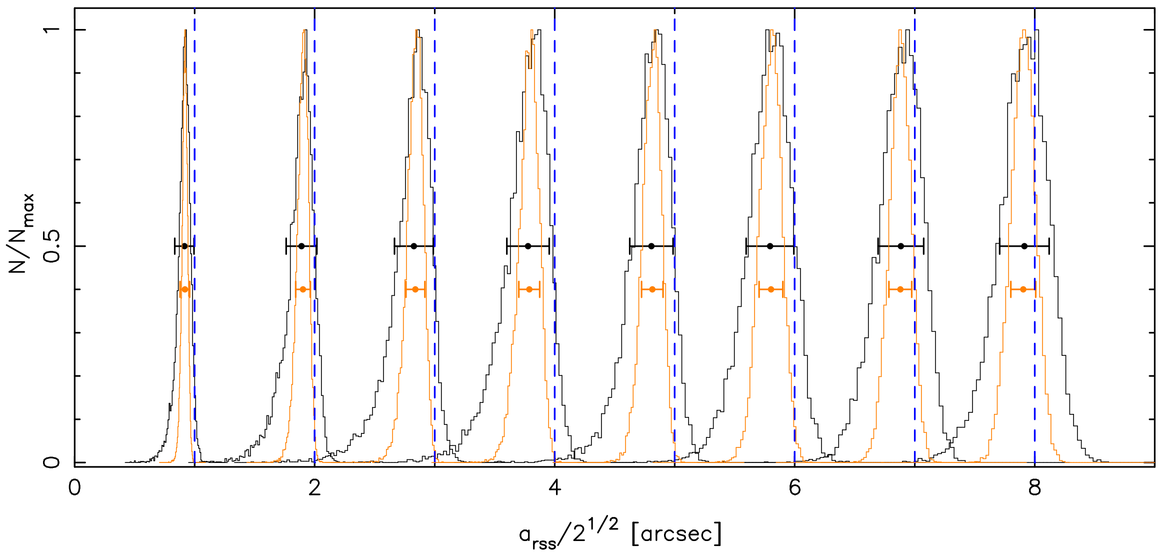

Fig. 1.— Distribution of variance-weighted mean intrinsic source size estimates. A dashed blue line indicates the known intrinsic size (σ_0) of each simulated circular Gaussian source. For each source size, a black (orange) histogram shows the distribution of size estimates derived using random samples of $N = 5$ ($N = 20$) sources. For each histogram, the median value of $\overline{a_{\rm rss}}/\sqrt{2}$ is plotted with an error bar indicating the median value of $\pm\overline{\Delta a_{\rm rss}}/\sqrt{2}$. The corresponding numerical values are given in Table 1.

Table 1. Median Source Size Errors[1]

	N=5			N=20		
σ_0	$\overline{a_{\rm rss}}/\sqrt{2}$	$\overline{\Delta a_{\rm rss}}/\sqrt{2}$	$\overline{a_{\rm rss}}/\sqrt{2}-\sigma_0$	$\overline{a_{\rm rss}}/\sqrt{2}$	$\overline{\Delta a_{\rm rss}}/\sqrt{2}$	$\overline{a_{\rm rss}}/\sqrt{2}-\sigma_0$
1	0.91	0.08	-0.09	0.92	0.04	-0.08
2	1.89	0.13	-0.11	1.90	0.06	-0.10
3	2.83	0.16	-0.17	2.84	0.08	-0.16
4	3.78	0.18	-0.22	3.79	0.09	-0.21
5	4.81	0.18	-0.19	4.81	0.09	-0.19
6	5.79	0.20	-0.21	5.80	0.10	-0.20
7	6.88	0.19	-0.12	6.88	0.10	-0.12
8	7.91	0.21	-0.09	7.90	0.10	-0.10

[1]All values are given in arcsec.

3. Testing

I tested the above procedure for determining $\overline{a_{\rm rss}}$ using a set of 28 *MARX* simulations originally produced to test the Mexican Hat Optimization (*MHO*) algorithm (Houck 2007). Each simulation contains a test pattern of circular Gaussian sources with intrinsic sizes $\sigma_0 = 1 - 8$ arcsec. Each test pattern fills the ACIS-I detector array. Five different exposure times are included, sampling a moderate range in source signal to noise ratio.

Using this test data, I performed two sets of Monte Carlo trials. For each input source size, σ_0, a sample of N sources were chosen at random. These randomly selected sources might have any of five different exposure times and might fall anywhere in the ACIS-I field of view. Their source ellipse parameters (σ_1, σ_2) were determined using the *MHO* algorithm. The corresponding PSF ellipse parameters (b_1, b_2) were obtained from the parameterization of Allen, Jerius & Gaetz (2004). For the each sample of N measurements, a mean intrinsic source size, $\overline{a_{\rm rss}} \pm \overline{\Delta a_{\rm rss}}$, was estimated using equations (9) and (10), assuming a 10% uncertainty in the PSF size. This process was repeated 10^4 times for $N = 5$ and for $N = 20$. The results are summarized in Figure 1 and Table 1.

As expected, increasing the sample size by a factor of four reduces the statistical error $(\overline{\Delta a_{\rm rss}})$ by about a factor of two. The systematic error in the estimated intrinsic source size $(\overline{a_{\rm rss}}/\sqrt{2} - \sigma_0)$ is dominated by the systematic error in the PSF ellipse parameters, b_i. This systematic error arises in part because the Allen, Jerius & Gaetz (2004) parameterization of the PSF is based on *SAOSAC* simulations, but the simulated data was generated by *MARX*.

4. Recommendations

1. Use equations (6) and (7) to estimate the intrinsic size of the source, $(a_{{\rm rss},i}/\sqrt{2}) \pm (\Delta a_{{\rm rss},i}/\sqrt{2})$, projected onto the tangent plane of each available observation. The factors of $1/\sqrt{2}$ ensure that, when applied to circular source images, the statistic value gives the radius of the source image. Ideally, the PSF size, b_i, should be computed using weights derived from the source spectrum and instrument response within the current energy band.

2. Use equations (9) and (10) to derive a minimum-variance estimate of the intrinsic source size, $\overline{a_{\rm rss}}$, and the corresponding uncertainty, $({\rm Var}[\overline{a_{\rm rss}}])^{1/2}$.

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A. Convolution of Two Elliptical Gaussians

In this section, I will use the convolution theorem to derive an expression for the convolution of two elliptical Gaussians. The convolution theorem is

$$\mathscr{F}[s \star p] = \mathscr{F}[s]\mathscr{F}[p],$$

where $\mathscr{F}[s]$ denotes the Fourier transform of a function s, defined as

$$S(u,v) = \mathscr{F}[s] \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(x,y)\, \mathrm{e}^{-i2\pi(ux+vy)}\, \mathrm{d}x\, \mathrm{d}y.$$

The inverse transform is

$$s(x,y) = \mathscr{F}^{-1}[S] \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(u,v)\, \mathrm{e}^{i2\pi(ux+vy)}\, \mathrm{d}u\, \mathrm{d}v.$$

The Fourier transform of an elliptical Gaussian with shape parameters $\{a_1, a_2, \phi\}$, centered at the origin, is

$$S(u,v) = \frac{s_0}{a_1 a_2} \iint \exp\left[-\pi(\mathcal{A}\mathbf{x})^2 - i2\pi\mathbf{u}\cdot\mathbf{x}\right] \mathrm{d}x\, \mathrm{d}y, \tag{A1}$$

where $\mathcal{A}$ is defined in equation (2). Changing variables so that $\mathbf{x} = \mathcal{A}^{-1}\mathbf{x}'$, we have

$$S(u,v) = \frac{s_0}{a_1 a_2} \iint \exp\left[-\pi\mathbf{x}'^2 - i2\pi\mathbf{u}\cdot(\mathcal{A}^{-1}\mathbf{x}')\right] \left|\frac{\partial(x,y)}{\partial(x',y')}\right| \mathrm{d}x'\, \mathrm{d}y'. \tag{A2}$$

From the definition of $\mathcal{A}$,

$$\mathbf{x} = \mathcal{A}^{-1}\mathbf{x}' = \mathcal{R}_\phi{}^{-1}\mathcal{U}^{-1}\mathbf{x}',$$

so that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x'a_1\cos\phi - y'a_2\sin\phi \\ x'a_1\sin\phi + y'a_2\cos\phi \end{pmatrix},$$

and the Jacobian determinant is

$$\left|\frac{\partial(x,y)}{\partial(x',y')}\right| = a_1 a_2.$$

Substituting in equation (A2) these expressions for $\mathcal{A}^{-1}\mathbf{x}'$ and the Jacobian determinant, we have

$$S(u',v') = s_0 \int_{-\infty}^{\infty} \exp\left[-\pi x'^2 - i2\pi x'u'\right] \mathrm{d}x' \int_{-\infty}^{\infty} \exp\left[-\pi y'^2 - i2\pi y'v'\right] \mathrm{d}y', \tag{A3}$$

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where we have defined

$$u' \equiv a_1(u\cos\phi + v\sin\phi), \tag{A4}$$

$$v' \equiv a_2(-u\sin\phi + v\cos\phi). \tag{A5}$$

Rewriting the x' integral in equation (A3) as

$$\int_{-\infty}^{\infty} \exp\left[-\pi x'^2 - i2\pi x'u'\right] \mathrm{d}x' = \mathrm{e}^{-\pi u'^2} \int_{-\infty}^{\infty} \exp\left[-\pi(x' + iu')^2\right] \mathrm{d}x', \tag{A6}$$

we can evaluate the definite integral on the right-hand side of equation (A6) by substituting $t \equiv x' + iu'$, and noting that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\pi t^2} \mathrm{d}t = 1.$$

It follows that

$$S(u', v') = s_0 \mathrm{e}^{-\pi(u'^2 + v'^2)},$$

where u' and v' are defined in equations (A4) and (A5).

For convenience in computing the convolution, we can choose a coordinate system so that the major axis of the PSF ellipse lies along the x-axis. In these coordinates, the PSF Gaussian has Fourier transform

$$P(u, v) = \mathscr{F}[p] = p_0 \exp\left[-\pi(b_1^2 u^2 + b_2^2 v^2)\right], \tag{A7}$$

and the source Gaussian has Fourier transform

$$S(u, v) = \mathscr{F}[s] = s_0 \exp\left\{-\pi\left[a_1^2(u\cos\alpha + v\sin\alpha)^2 + a_2^2(-u\sin\alpha + v\cos\alpha)^2\right]\right\}, \tag{A8}$$

where $\alpha \equiv \phi - \psi$ is the angle between the major axis of the source ellipse and the major axis of the PSF ellipse. Applying the convolution theorem, the PSF-convolved source is $c(x, y) \equiv s \star p$, which can now be written in the form

$$c(x, y) = \mathscr{F}^{-1}[S(u, v)P(u, v)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(u, v)\, P(u, v)\, \mathrm{e}^{i2\pi(ux + vy)}\, \mathrm{d}u\, \mathrm{d}v. \tag{A9}$$

Substituting equations (A7) and (A8) into equation (A9), we obtain

$$c(x, y) = s_0 p_0 \iint \exp\left\{-\pi\left[Au^2 + Buv + Cv^2\right] + i2\pi(ux + vy)\right\} \mathrm{d}u\, \mathrm{d}v.$$

where

$$A \equiv a_1^2 + b_1^2\cos^2\alpha + b_2^2\sin^2\alpha,$$
$$C \equiv a_2^2 + b_1^2\sin^2\alpha + b_2^2\cos^2\alpha,$$
$$B \equiv 2\,(b_1^2 - b_2^2)\,\sin\alpha\cos\alpha.$$

If $\alpha = n\pi/2$ or if $b_1 = b_2$, then $B = 0$ and the uv cross-term vanishes. In the general case, for $b_1 \neq b_2$, the cross-term can be eliminated by introducing a coordinate rotation. To eliminate the cross-term, introduce new coordinates, $\mathbf{u} = \mathcal{R}_\delta \mathbf{u}'$, defined by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix},$$

and then choose the rotation angle δ so that the new $u'v'$ cross-term vanishes. Working through the algebra, it follows that the cross-term vanishes for

$$\delta = \frac{1}{2}\tan^{-1}\frac{B}{C-A}.$$

In these coordinates, equation (A9) becomes

$$c(x,y) = s_0 p_0 \iint \exp\left[-\pi\sigma_1^2 u'^2 - \pi\sigma_2^2 v'^2 + i2\pi(\mathcal{R}_\delta \mathbf{u}')\cdot\mathbf{x}\right] \mathrm{d}u'\,\mathrm{d}v', \tag{A10}$$

where

$$\sigma_1^2 \equiv A\cos^2\delta - B\sin\delta\cos\delta + C\sin^2\delta, \tag{A11}$$

$$\sigma_2^2 \equiv A\sin^2\delta + B\sin\delta\cos\delta + C\cos^2\delta. \tag{A12}$$

Performing the integration in equation (A10) using the same techniques that were applied to equation (A1), it follows that

$$c(x,y) = \frac{s_0 p_0}{\sigma_1\sigma_2}\exp\left\{-\pi\left[\frac{1}{\sigma_1^2}(x\cos\delta - y\sin\delta)^2 + \frac{1}{\sigma_2^2}(x\sin\delta + y\cos\delta)^2\right]\right\}, \tag{A13}$$

for the case $b_1 \neq b_2$. The simpler case with $b_1 = b_2$ is left as an exercise for the reader. This result demonstrates that the convolution of two elliptical Gaussians yields an elliptical Gaussian.

Note that $\delta \neq \phi$; in general, the PSF-convolved ellipse does not have the same orientation as the source ellipse. Expressing δ in terms of the various ellipse parameters, we have

$$\tan 2\delta = \frac{\tan 2\alpha}{1+\xi\sec 2\alpha}, \quad \text{where } \xi \equiv \frac{a_1^2 - a_2^2}{b_1^2 - b_2^2}. \tag{A14}$$

Clearly, the orientation of the PSF-convolved ellipse differs from that of the source ellipse whenever the source and PSF are misaligned (non-circular) ellipses.

The identity

$$\sigma_1^2 + \sigma_2^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2, \tag{A15}$$

follows trivially from the definition of σ_1^2 and σ_2^2 and is valid for the general case of the convolution of two elliptical Gaussians with arbitrary orientation.

Other special cases follow immediately:

$$\begin{aligned}
\alpha = 0, \pi &\implies \sigma_1^2 = a_1^2 + b_1^2, \quad \sigma_2^2 = a_2^2 + b_2^2. \\
\alpha = \pi/2 &\implies \sigma_1^2 = a_1^2 + b_2^2, \quad \sigma_2^2 = a_2^2 + b_1^2. \\
b_1 = b_2 = b &\implies \sigma_i^2 = a_i^2 + b^2.
\end{aligned}$$