A Statistic for a Crater Detection Algorithm

Version 0.1-0

John E. Davis
<davis@space.mit.edu>

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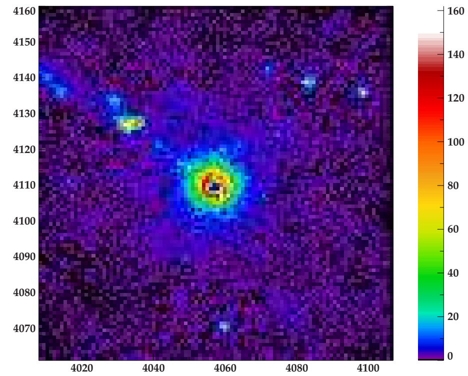
1 Overview

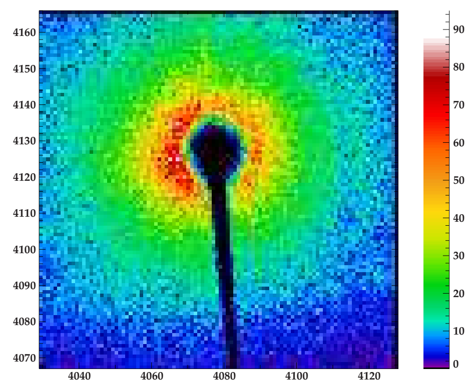
The purpose of this document is to present a statistic that may be used as the basis for an algorithm to detect the so-called pile-up craters created by photon pile-up from a bright X-ray source. The standard level-3 source detection pipeline misses such sources and instead incorrectly identifies sources around the rim of the crater. Knowing the locations and sizes of the craters will allow such false detections to be excluded.

Craters come in various sizes and shapes. A example of a standard looking crater is shown in figure 1a. Because of serial CTI, some craters look more like canyons as the example in 1b illustrates. If the cratering source experiences flares, something more akin to a "moat" can result; such an example is show in figure 1c.

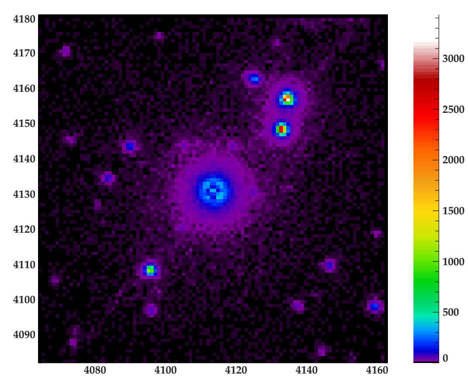
In the next section, a simple mathematical characterization of craters based upon moments is presented. From these considerations, a statistic that may be used for crater-detection is proposed. Some constraint equations used to filter out candidate regions and limit the number of false detections are given in section 3. Fol-



(a) Standard



(b) Canyon



(c) Moat

Figure 1: Figure showing images of the three types of craters described in the text.

lowing a brief summary is an appendix where a crude estimate of the minimum count rate for a crater is given, as well some software implementation notes.

2 General Considerations

Suppose that an image is made by binning the events using some pixel grid expressed in aspect-corrected sky coordinates. Then consider a region Ω of radius R that contains a standard crater at its center. Such a region is expected to have only a few counts M_0 at its center with most counts uniformly azimuthally distributed outside some core of radius r. Let M denote the total number of counts contained in the region, and let M_0 be the number contained in the core. Similarly, let $M_1 = M - M_0$ be the number outside the core. Now erect a Cartesian coordinate system with the origin at the center of the region. Then each image pixel will have some coordinate (x, y) and contain m_{xy} counts so that

$$M = \sum_{xy \in \Omega} m_{xy}. \tag{1}$$

Now consider the following moments of the count distribution in the region:

$$I_x = \sum_{xy \in \Omega} m_{xy} x \tag{2}$$

$$I_y = \sum_{xy \in \Omega} m_{xy} y \tag{3}$$

$$I_{xy} = \sum_{xy \in \Omega} m_{xy} xy \tag{4}$$

$$I'_{xy} = \sum_{xy \in \Omega} m_{xy} |x||y| \tag{5}$$

$$I_{xx} = \sum_{xy \in \Omega} m_{xy} x^2 \tag{6}$$

$$I_{yy} = \sum_{xy \in \Omega} m_{xy} y^2 \tag{7}$$

For an approximately azimuthally symmetric crater, it is expected that

$$I_x \approx I_y \approx 0 \tag{8}$$

and

$$I_{xy} \approx 0. \tag{9}$$

On the other hand, since the crater is expected to have few counts near the origin, $I_{xx} \approx I_{yy}$ should be relatively large compared to the moments for a region that contains an equal number of counts uniformly spread over Ω.

If I is defined to be $I_{xx} + I_{yy}$, then the radius of the crater may be characterized as

$$r = \sqrt{I/M}. \tag{10}$$

Furthermore, for an azimuthally symmetric distribution, I can be related to I'_{xy} as follows. In the continuum limit, I may be written as

$$I = 2\pi \int_0^R \mathrm{d}r \rho(r) r^3, \tag{11}$$

where $\rho(r)$ represents the count-density at the radial coordinate r. Similarly,

$$I'_{xy} = \int_0^R r\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \rho(r) |r\cos\theta||r\sin\theta|. \tag{12}$$

The integral over θ may be easily carried out to yield the relation

$$I = \pi I'_{xy}. \tag{13}$$

With the above observations in mind, consider the quantity T defined by

$$T = \frac{2I_2 - k_1(I_1^2/M + |I_{xy}|) - k_2|I - \pi I'_{xy}| - k_3 I_3}{NR^2 M_0/M_1}, \tag{14}$$

where

$$I_1 = \max(|I_x|, |I_y|), \tag{15}$$

$$I_2 = \min(I_{xx}, I_{yy}), \tag{16}$$

$$I_3 = (1 + m_{00})\pi R^4/2. \tag{17}$$

Here k_1, k_2, and k_3 are positive constants, and N is the number of CCD frames that contributed to the image, which serves to normalize T.

For an azimuthally symmetric crater, the first term in the numerator of equation (14) will be much larger than the remaining terms; hence this quantity is expected to be a large positive value. For the reasons outlined above, the terms involving k_1 and k_2 vanish for a perfectly symmetric distribution; hence they are expected to be quite small for a cratered source.

The last term, which involves the number of counts m_{00} in the central pixel, will be small for a cratered source where most of the counts are distributed away from the center. However, for a non-cratered source m_{00} will be quite large making this term very negative and most likely the most dominant term in the numerator. Hence for a non-cratered point source at the center of the region, T will most likely be negative.

Now consider the effect of the last term for a region Ω that contains mainly background events. In this case, the value of m_{00} will be representative of all the other values of m_{xy} in the region. Since there are πR^2 pixels in the region, $m_{00}\pi R^2$ will be a value that is approximately equal the observed number of counts M in the region. In this scenario the value of the last term will be something like $MR^2/2$. For this reason, the I_3 may be regarded as the background contribution to I and as such,

$$r = \sqrt{(I - I_3)/M} \tag{18}$$

can be taken as the definition of the radius of the crater instead of equation (10).

The above considerations imply that the numerator of equation (14) will be relatively large and positive when the region contains a crater at its center, and will be relatively small or negative otherwise. The denominator, which is proportional to the ratio of the number of counts in the core to that outside the core, will tend to further increase the magnitude of T when the region contains a crater.

For the above reasons, T may be regarded as a statistic that may be used to test for the presence of a crater at center of a region. If T is large and positive, then the region is likely to contain a crater at its center. If T is small or negative, then the region is unlikely to contain a crater.

3 Region Mask

There are some problems with the simple statistic T given by equation (14) of the previous section. One problem arises for bright background regions that contain many counts, such as regions in the scattering wings or halo of a bright source. Here statistical fluctuations in the count distribution could result in cases where T is quite large. This section describes a *mask* that may be used to filter out such regions.

For a large enough region centered upon a crater, the region should contain a minimum number of counts. Call this number $M_{\min}$. Then the first part of the mask may be expressed in the form

$$M \geq M_{\min}. \tag{19}$$

An estimate of $M_{\min}$ is given in the appendix.

The second part of the mask constrains the center of the crater to be at the center of the region. From equations 2 and 3, the distance from the center of the region to the center of the crater can be taken to be $\sqrt{I_x^2 + I_y^2}/M$. Hence the constraint that the center of the crater should be less than one pixel from the region center can be written as the mask

$$\sqrt{I_x^2 + I_y^2} \leq k_4 M. \tag{20}$$

It is easy to imagine that the above two constraints can be satisfied by a uniformly distributed count distribution provided that M is large enough. What is needed is a constraint that masks out uniformly distributed regions. Such a constraint can be formulated in mathematical terms as follows.

Suppose that the region Ω is partitioned into two subregions with areas A_1 and A_2, and with mean counts per pixel $\bar{m}_1$ and $\bar{m}_2$, respectively. The mean count density of the combined region is given by

$$\bar{m} = \frac{A_1 \bar{m}_1 + A_2 \bar{m}_2}{A}, \tag{21}$$

where $A = A_1 + A_2$. The variance Var$[m]$ in the mean count density for combined region is defined to be

$$\text{Var}[m] = \frac{1}{A}\sum_{ij\in A_1}(m_{ij} - \bar{m})^2 + \frac{1}{A}\sum_{ij\in A_2}(m_{ij} - \bar{m})^2, \tag{22}$$

which can be written in terms of the individual subregion variances as

$$\text{Var}[m] = \frac{A_1}{A}\text{Var}[m_1] + \frac{A_2}{A}\text{Var}[m_2] + \frac{A_1 A_2}{A^2}(\bar{m}_1 - \bar{m}_2)^2. \tag{23}$$

If the counts in the subregions are Poisson distributed, then Var$[m_1] \approx \bar{m}_1$ and Var$[m_2] \approx \bar{m}_2$, with the result

$$\text{Var}[m] \approx \bar{m} + \frac{A_1 A_2}{A^2}(\bar{m}_2 - \bar{m}_1)^2. \tag{24}$$

If both the subregions have the same underlying Poisson distribution, then the term involving the difference in the observed means $|\bar{m}_2 - \bar{m}_1|$ can be neglected leaving Var$[m] \approx \bar{m}$. In other words, a region consisting of a uniform distribution of Poisson-distributed counts will have a variance of the average count density. However, if the region is not uniformly distributed, then the last term, which scales as the square of the difference $\bar{m}_2 - \bar{m}_1$, can be much larger than $\bar{m}$. Hence, a mask that picks out non-uniformly distributed regions can be written as

$$\text{Var}[m] > \bar{m} + k_5\bar{m}^2, \tag{25}$$

where k_5 is a small non-zero constant.

4 Summary

In this work, a static for use in a pile-up crater detection algorithm was proposed in the form of equation (14). To cut down on the number of false detections, the statistic should only be applied to regions of the image that satisfy the set of constraints given in equations 19, 20, and 25. These equations depend upon a number of fixed parameters, whose suggested values are given in the following table:

k_1	4
k_2	1
k_3	0.5
k_4	1.5
k_5	0.25
$M_{\min}$	0.35
R	9
r	2.5

These values were empirically determined via trial and error, and as such are most likely subject to change in a future update.

A An estimate of $M_{\min}$

A crater will start to form when the observed count rate outside the core of the PSF becomes comparable with the count rate in the core. Further increases in the source flux will increase the count rate outside the core and decrease the rate in the core. Let f denote the average number of charge-clouds per frame in the CCD from a point source, and let x be the fraction that is contained in the core. Then the per frame count-rate M_0 in the core may be estimated by

$$M_0 = \sum_{k\geq 1} \alpha^{k-1} \frac{(xf)^k}{k!} e^{-xf} \tag{26}$$

$$= e^{-xf} \frac{1}{\alpha} \sum_{k\geq 1} \frac{(\alpha x f)^k}{k!} \tag{27}$$

$$= e^{-xf} \frac{1}{\alpha} (e^{\alpha x f} - 1) \tag{28}$$

where α is the probability that two overlapping charge clouds will give rise to an event. Assume that there are n independent detection cells outside the core. Then it is straightforward to show that the expected per-frame count rate outside the core is given by

$$M_1 = n e^{-(1-x)f/n} \frac{1}{\alpha} (e^{\alpha(1-x)f/n} - 1). \tag{29}$$

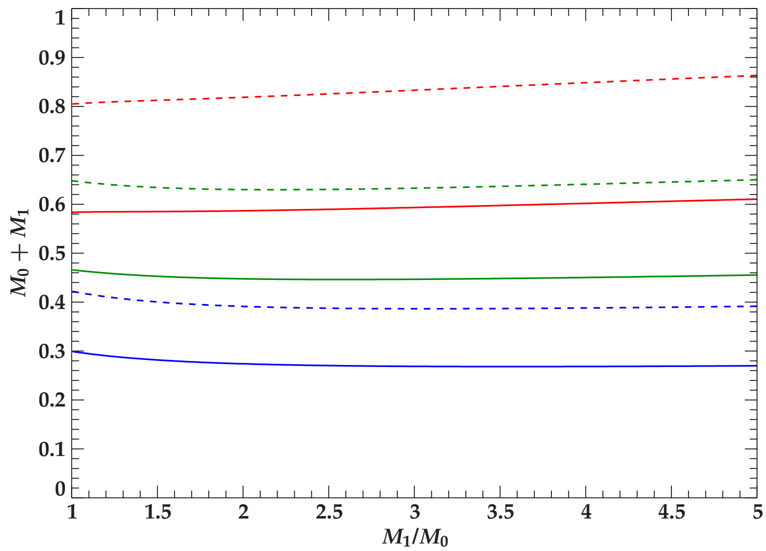


Figure 2: Figure showing a plot of the ratio of the number of counts per frame outside the core to those in the core vs the total number of counts per frame for various values of x and α. The solid curves correspond to $\alpha = 0$ and the dashed curves are for $\alpha = 0.5$. The red, green, and blue curves correspond to values of x equal to 0.85, 0.9, and 0.95, respectively.

Cratering will occur when $M_1/M_0 > 1$. Figure 2 shows a plot of M_1/M_0 vs $M = M_0 + M_1$ for various values of α and x for $n = 3$. The figure shows that the smallest value of M occurs for $\alpha = 0$, and $f = 0.95$, where M is a bit less than 0.3. Since this value of α is a bit unrealistic, a value of $M_{\rm min} = 0.35$ was felt to be a reasonable compromise.

B Implementation Notes

A naive computation of T as given by equation (14) is rather straight-forward. For example, here is pseudo-code that computes the moment I_y that contributes to T for the region centered at i_0, j_0:

```
iR = (int)(R+0.5);      /* round to nearest integer */
```

```
Iy = 0;
for (i = i0-iR; i <= i0+iR; i++)
  {
     if ((i < 0) || (i >= nx)) continue;
     for (j = j0-iR; j <= j0+iR; j++)
       {
          if ((j < 0) || (j >= ny)) continue;
          if ((j-j0)^2 + (i-i0)^2 > R^2) continue;
          I_y += (j-j0)*m[i,j];
       }
  }
```

However for values of R greater than about 3, it is much better to cast the computation in terms of correlation integrals and make use of FFTs compute the correlations. See the **S-Lang** reference implementation for an example of this approach.