notes to self

Background marginalized X-ray source intensity

Vinay L. Kashyap

Harvard-Smithsonian Center for Astrophysics, 60 Garden Street, Cambridge, MA 02138

vkashyap@cfa.harvard.edu

ABSTRACT

The posterior probability distribution of the source intensity is calculated for the case when imperfect apertures are used to measure counts.

Subject headings: methods: data analysis – methods: statistical

1. Introduction

2. X-ray Aperture photometry

Suppose C counts are observed in an "source" aperture of area A_S and B counts are observed in a "background" aperture of area A_B. Further suppose that the source aperture encloses a fraction f of a PSF centered in A_S and spills over by a fraction g over the background aperture. Note that these areas need not be circular, concentric, or even contiguous.

The observed counts are presumed to be generated via a Poisson process, i.e.,

$$\begin{aligned} C \sim Pois(\theta) &\equiv Pois(fs+b) \\ B \sim Pois(\phi) &\equiv Pois(gs+rb)\,, \end{aligned} \tag{1}$$

where θ, ϕ are the Poisson intensities that lead to the observations in the source and background apertures respectively, s is the intensity of the source, and b is the intensity of the background normalized to the area of the source aperture, and $r = \frac{A_B}{A_S}$ is the ratio of the background and source apertures.

We seek to determine the posterior probability of s, marginalized over the background,

$$p(s|CB) = \int dbp(sb|CB)\,, \tag{2}$$

where the integrand on the rhs is the joint posterior probability distribution of s and b.

2.1. The Classical Case

In the high counts regime, we can approximate s and b by their MLE values, and thus write

$$\begin{aligned} C &= fs+b \\ B &= gs+rb \end{aligned} \tag{3}$$

which leads to the solution

$$\begin{aligned} s &= \frac{rC-B}{rf-g} \\ b &= \frac{fB-gC}{rf-g}\,, \end{aligned} \tag{4}$$

with errors propagated under an assumption of Gaussian regime,

$$\begin{aligned} \sigma^2(s) &= \frac{r^2C+B}{(rf-g)^2} \\ \sigma^2(b) &= \frac{f^2B+g^2C}{(rf-g)^2}\,. \end{aligned} \tag{5}$$

2.2. The Bayesian Calculation

First note that the variable pairs (θ,ϕ) and (s,b) are linear transforms of each other. Thus,

$$\begin{aligned} p((\theta\phi|CB)d\theta d\phi &= p(sb|CB)J(\theta,\phi;s,b)dsdb \\ &= p(sb|CB)\left|\frac{\partial(\theta,\phi)}{\partial(s,b)}\right|dsdb \\ &= p(sb|CB)(rf-g)dsdb\,. \end{aligned} \tag{6}$$

Now, the joint posterior probability density $p(\theta\phi|CB)$ can be written out using Bayes' Theorem as

$$\begin{aligned} p(\theta\phi|CB) &= \frac{p(\theta)p(\phi)p(C|\theta\phi)p(B|\theta\phi)}{p(CB)} \\ &\equiv \frac{p(\theta)p(\phi)p(C|\theta)p(B|\phi)}{p(CB)}\,, \end{aligned} \tag{7}$$

where the likelihoods are simplified because C and B are independent measurements; $p(\theta)$ and $p(\phi)$ are priors on the Poisson intensity in the source and background aperture, and we use γ-functions for them,

$$p(\theta) = \frac{{\beta_S}^{\alpha_S}\theta^{\alpha_S-1}\ e^{-\beta_S\theta}}{\Gamma(\alpha_S)}\,, \tag{8}$$

$$p(\phi) = \frac{{\beta_B}^{\alpha_B}\phi^{\alpha_B-1}\ e^{-\beta_B\phi}}{\Gamma(\alpha_B)}\,, \tag{9}$$

where $\alpha_S,\alpha_B,\beta_S,\beta_B$ are parameters that define the shape of the functions. For non-informative priors, a typical choice is $\alpha_S=\alpha_B=1$ and $\beta_S=\beta_B=0$, representing the information encoded in an observation of 0 counts obtained over 0 area.[1] For the sake of simplicity, we require that α_S,α_B be integers. This is not restrictive, because priors can always be set in terms of expected counts (α_S,α_B =expected counts+1).

The likelihoods are

$$p(C|\theta) = \frac{\theta^C\ e^{-\theta}}{\Gamma(C+1)} \tag{10}$$

$$p(B|\phi) = \frac{\phi^B\ e^{-\phi}}{\Gamma(B+1)}\,. \tag{11}$$

$$\tag{12}$$

[1]Note that this leads to an improper prior, i.e., one that is not normalizable. However, use of such priors does not imply that the posterior densities are also improper.

The joint posterior (Equation 7) can then be written as

$$p(\theta\phi|CB) = \theta^{C+\alpha_S-1}\phi^{B+\alpha_B-1}\ e^{-\theta(1+\beta_S)-\phi(1+\beta_B)} \times \frac{1}{p(CB)}\frac{{\beta_S}^{\alpha_S}{\beta_B}^{\alpha_B}}{\Gamma(\alpha_S)\Gamma(\alpha_B)\Gamma(C+1)\Gamma(B+1)}, \quad (13)$$

where the normalization factor $p(CB)$ is determined by requiring that

$$\int_0^\infty d\theta \int_0^\infty d\phi\ p(\theta\phi|CB) = 1\,. \quad (14)$$

Considering that

$$\int_0^\infty dx\ x^{A-1}\ e^{-xB} = \frac{\Gamma(A)}{B^A}\,, \quad (15)$$

we have

$$\int_0^\infty d\theta\ \theta^{C+\alpha_S-1}\ e^{-\theta(1+\beta_S)} = \frac{\Gamma(C+\alpha_S)}{(1+\beta_S)^{C+\alpha_S}}\,, \quad \text{and} \quad (16)$$

$$\int_0^\infty d\phi\ \phi^{B+\alpha_B-1}\ e^{-\phi(1+\beta_B)} = \frac{\Gamma(B+\alpha_B)}{(1+\beta_B)^{B+\alpha_B}}\,. \quad (17)$$

Therefore, by Equations 13 and 14,

$$p(CB) = \frac{\Gamma(C+\alpha_S)\Gamma(B+\alpha_B){\beta_S}^{\alpha_S}{\beta_B}^{\alpha_B}}{\Gamma(\alpha_S)\Gamma(\alpha_B)\Gamma(C+1)\Gamma(B+1)(1+\beta_S)^{C+\alpha_S}(1+\beta_B)^{B+\alpha_B}}\,. \quad (18)$$

(Note that for $\alpha_S = \alpha_B = 1$ and $\beta_S = \beta_B = 0$, the determinate portion of the above expression reduces to $p(CB) = 1$.) With this normalization, Equation 13 reduces to

$$p(\theta\phi|CB) = \frac{(1+\beta_S)^{C+\alpha_S}(1+\beta_B)^{B+\alpha_B}}{\Gamma(C+\alpha_S)\Gamma(B+\alpha_B)}\ \theta^{C+\alpha_S-1}\phi^{B+\alpha_B-1}\ e^{-\theta(1+\beta_S)-\phi(1+\beta_B)}\,. \quad (19)$$

We change variables from $\theta, \phi \to s, b$ (see Equation 6) and write

$$\begin{aligned} p(\theta\phi)\ d\theta\ d\phi &= (rf-g)\ p(\theta(s,b)\phi(s,b)|CB)\ ds\ db \\ &= ds\ db\ (rf-g)\frac{(1+\beta_S)^{C+\alpha_S}(1+\beta_B)^{B+\alpha_B}}{\Gamma(C+\alpha_S)\Gamma(B+\alpha_B)} \times \\ &\quad (fs+b)^{C+\alpha_S-1}(gs+rb)^{B+\alpha_B-1}\ e^{-(fs+b)(1+\beta_S)-(gs+rb)(1+\beta_B)} \\ &= ds\ db\ (rf-g)\frac{(1+\beta_S)^{C+\alpha_S}(1+\beta_B)^{B+\alpha_B}}{\Gamma(C+\alpha_S)\Gamma(B+\alpha_B)} \times \\ &\quad (fs+b)^{C+\alpha_S-1}(gs+rb)^{B+\alpha_B-1}\ e^{-s(f+g+f\beta_S+g\beta_B)}\ e^{-b(1+r+\beta_S+r\beta_B)}\,. \end{aligned} \quad (20)$$

Thus, the posterior probability of the source intensity,

$$
\begin{aligned}
p(s|CB)ds &= ds \int_0^\infty db\ p(sb|CB) \\
&\equiv ds\ (rf-g)\frac{(1+\beta_S)^{C+\alpha_S}(1+\beta_B)^{B+\alpha_B}}{\Gamma(C+\alpha_S)\Gamma(B+\alpha_B)} \times\ e^{-s(f+g+f\beta_S+g\beta_B)} \\
&\quad \int_0^\infty db\ (fs+b)^{C+\alpha_S-1}(gs+rb)^{B+\alpha_B-1}\ e^{-b(1+r+\beta_S+r\beta_B)}\,. \qquad (21)
\end{aligned}
$$

Now, we can write

$$
(x+y)^n = \sum_{k=0}^{n} {}^nC_k x^k y^{n-k} = \sum_{k=0}^{n} \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} x^k y^{n-k}
$$

and thus expanding the factors within the integrand with $C+\alpha_S-1 = N$ and $B+\alpha_B-1 = M$ (this is where it is necessary to assume that α_S and α_B are integers),

$$
\begin{aligned}
(fs+b)^{C+\alpha_S-1} &= (fs+b)^N \\
&= \sum_{k=0}^{N} f^k\ \frac{\Gamma(N+1)}{\Gamma(k+1)\Gamma(N-k+1)} s^k b^{N-k}\,, \qquad (22) \\
(gs+rb)^{B+\alpha_B-1} &= (gs+rb)^M \\
&= \sum_{j=0}^{M} g^j\ r^{M-j}\ \frac{\Gamma(M+1)}{\Gamma(j+1)\Gamma(M-j+1)} s^j b^{M-j}\,. \qquad (23)
\end{aligned}
$$

Thus, the integrand in Equation 21 can be written as

$$
\begin{aligned}
&\int_0^\infty db\ (fs+b)^{C+\alpha_S-1}(gs+rb)^{B+\alpha_B-1}\ e^{-b(1+r+\beta_S+r\beta_B)} \\
&= \sum_{k=0}^{N}\sum_{j=0}^{M} f^k\ g^j\ r^{M-j}\ s^{k+j}\ \frac{\Gamma(N+1)\Gamma(M+1)}{\Gamma(k+1)\Gamma(N-k+1)\Gamma(j+1)\Gamma(M-j+1)} \times \\
&\quad \int_0^\infty db\ b^{N+M-k-j}\ e^{-b(1+r+\beta_S+r\beta_B)}\,, \\
&\equiv \sum_{k=0}^{N}\sum_{j=0}^{M} f^k\ g^j\ r^{M-j}\ s^{k+j} \times \\
&\quad \frac{\Gamma(N+1)\Gamma(M+1)\Gamma(N+M-k-j+1)}{\Gamma(k+1)\Gamma(N-k+1)\Gamma(j+1)\Gamma(M-j+1)(1+r+\beta_S+r\beta_B)^{N+M-k-j+1}}\,. \qquad (24)
\end{aligned}
$$

Putting it all together,

$$
p(s|CB)ds = ds \int_0^\infty db\ p(sb|CB)
$$

$$
\begin{aligned}
&\equiv \; ds\ (rf-g)\frac{(1+\beta_S)^{C+\alpha_S}(1+\beta_B)^{B+\alpha_B}}{\Gamma(C+\alpha_S)\Gamma(B+\alpha_B)} \times \\
&\quad \sum_{k=0}^{N}\sum_{j=0}^{M}(f^k\ g^j\ r^{M-j}\ s^{k+j}\ e^{-s(f+g+f\beta_S+g\beta_B)} \times \\
&\quad \frac{\Gamma(N+1)\Gamma(M+1)\Gamma(N+M-k-j+1)}{\Gamma(k+1)\Gamma(N-k+1)\Gamma(j+1)\Gamma(M-j+1)(1+r+\beta_S+r\beta_B)^{N+M-k-j+1}}) \,. \qquad (25)
\end{aligned}
$$

For the non-informative priors, $\alpha_S, \alpha_B = 1$ and $\beta_S = \beta_B = 0$,

$$
\begin{aligned}
p(s|CB)ds &= ds\ (rf-g)\frac{1}{\Gamma(C+1)\Gamma(B+1)} \times \\
&\quad \sum_{k=0}^{C}\sum_{j=0}^{B}(f^k\ g^j\ r^{B-j}\ s^{k+j}\ e^{-s(f+g)} \times \\
&\quad \frac{\Gamma(C+1)\Gamma(B+1)\Gamma(C+B-k-j+1)}{\Gamma(k+1)\Gamma(C-k+1)\Gamma(j+1)\Gamma(B-j+1)(1+r)^{C+B-k-j+1}}) \,. \qquad (26)
\end{aligned}
$$

And further, when the source PSF has no overlap with the background aperture, $g = 0$, and only the $j = 0$ term remains from the summation,

$$
\begin{aligned}
p(s|CB)ds &= ds\ \frac{1}{\Gamma(C+1)\Gamma(B+1)} \times \\
&\quad \sum_{k=0}^{C}(f^{k+1}\ r^{B+1}\ s^k\ e^{-sf} \times \\
&\quad \frac{\Gamma(C+1)\Gamma(B+1)\Gamma(C+B-k+1)}{\Gamma(k+1)\Gamma(C-k+1)\Gamma(B+1)(1+r)^{C+B-k+1}}) \,. \qquad (27)
\end{aligned}
$$

REFERENCES

Loredo, T.J. 1990, in "Maximum Entropy and Bayesian Methods", ed. P.F. Fougere (Dordrecht:Kluwer), 81

van Dyk, D.A., Connors, A., Kashyap, V.L., & Siemiginowska, A. 2001, ApJ, 548, 224

http://hea-www.harvard.edu/~kashyap/scar.html#PPD_SRC

This preprint was prepared with the AAS LaTeX macros v5.2.