

BsplineEval

A C-library to evaluate a spline from its B-spline representation.
edition 1.1.4 for BsplineEval version 1.1.4
25 January 2010

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1 Copying

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2 Introduction

BsplineEval This library implements the algorithms to evaluate a spline from its B-spline representation as described in the paper:

The Numerical Evaluation of a Spline from its B-Spline Representation by M. G. Cox. J. Inst. Maths. Applics (1978) 21, 135-143.

The **Algorithm A**, **Algorithm B** and **Algorithm C** as described in said paper are implemented. Moreover, the **algorithm C** with its associated derivatives and an optimized version of **algorithm C** with its derivatives are provided.

The functions **Bspline_Algo_a** and **Bspline_Algo_b** are the one dimensional spline evaluation from its B-spline representation using algorithm A and B, respectively.

The evaluation of $s(\theta, z)$, where θ is periodic, from its normalized B-spline representation are implemented in the functions: **Bspline_Algo_c**, **Bspline_Algo_c_derivs**, **Bspline_eval** and **Bspline_eval_derivs**.

The functions **Bspline_Algo_c** and **Bspline_Algo_c_derivs** allow the user to set the underlying algorithms used (algorithm 'a' or 'b'). **Bspline_eval** and **Bspline_eval_derivs** are the optimized versions of **Bspline_Algo_c** and **Bspline_Algo_c_derivs**, respectively.

Functions with a *_derivs* suffix also evaluates the derivatives with respect to θ and z . The results of the B-spline evaluation and its derivatives are stored in the structure:

```
typedef struct {
    double s                               /* s( theta, z ) */;
    double dsdz                            /* ds( theta, z ) / dz */;
    double dsdt                            /* ds( theta, t ) */;
} BsplineResult;
```


3 Example

3.1 A sample C program

```

#include <stdio.h>
#include <stdlib.h>

#include <bsplineval.h>
#include <spline_coef.h>

/*
   This program illustrates the sequences of calls to make to evaluate a
   spline.
*/

int use_selected_Bspline_evaluator( BsplineInput* bspline_input ) {

    double tt = 3.14, zz = 0.0;
    BsplineResult result;

    /* Algorithm B then Algorithm A will be used in the call to Algorithm C */
    bspline_input->method1 = 'b';
    bspline_input->method2 = 'a';

    if ( Bspline_Success !=
        BsplineAlgo_c_derivs( tt, zz, bspline_input, &result ) ) {
        fprintf( stderr,
            "BsplineAlgo_c_derivs returned with an error t=%.14f, "
            "z=%.14f\n", tt, zz );
        return EXIT_FAILURE;
    }

    return EXIT_SUCCESS;
}

/* This is the recommended used of the call to the library */
int use_optimized_Bspline( BsplineInput* bspline_input ) {

    double tt = 3.14, zz = 0.0;
    BsplineResult result;

    if ( Bspline_Success !=
        Bspline_eval_derivs( tt, zz, bspline_input, &result ) ) {
        fprintf( stderr,
            "Bspline_eval_derivs returned with an error t=%.14f, "

```

```

        "z=%.14f\n", tt, zz );
    return EXIT_FAILURE;
}

return EXIT_SUCCESS;

}

int main( int argc, char** argv ) {

    /* Common c; */
    BsplineInput* bspline_input = NULL;

    /*
     * The user is responsible for supplying the information necessary to
     * evaluate the B-spline: the knots in theta and z, the order of knots etc..
     * This structure must be filled in by the user before making any more calls
     * to the library.
     */
    spline_coef scoef;

    /*
     * It is the user's responsibility to read in the Bspline coefficients.
     * read_spline_data_( c->input, &input_unit, &c->rms_amplitude );
     */

    /* Allocate the memory used by the library */
    bspline_input = (BsplineInput*) Bspline_init( &scoef );
    if ( NULL == bspline_input ) {
        fprintf( stderr,
            "process( ) : Unable to allocate memory for eval_spline\n" );
        return EXIT_FAILURE;
    }

    /* This is the recommended used of the call to the library */
    if ( EXIT_FAILURE == use_optimized_Bspline( bspline_input ) ) {
        Bspline_free( bspline_input );
        return EXIT_FAILURE;
    }

    /* If the user insists on using one of the slow routines */
    if ( EXIT_FAILURE == use_selected_Bspline_evaluator( bspline_input ) ) {
        Bspline_free( bspline_input );
        return EXIT_FAILURE;
    }

    /* Must free memory which was allocated in the call to Bspline_init */

```

```
    Bspline_free( bspline_input );  
    return EXIT_SUCCESS;  
}
```


4 Library Routines

4.1 BsplineAlgo_a

BsplineAlgo_a – The evaluation of $s(x)$ from its normalized B-spline representation using Scheme A (repeated convex combinations).

Synopsis

```
#include <BsplineEval/bsplineeval.h>
#include <BsplineEval/spline_coef.h>

BsplineErr BsplineAlgo_a(
    double x,
    size_t j_index,
    size_t order,
    double *c,
    double *d,
    double *knot,
    double *result
);
```

Parameters

```
double x    The point to evaluate the B-spline at

size_t j_index
            knot[ j_index-1 ] <= x < knot[ j_index ], j

size_t order
            the order of the B-spline, n

double *c   the B-spline coefficients, subscripted c

double *d   work space of dim : number of knots + order

double *knot
            the B-spline knots, subscripted x

double *result
            s(x)
```

Description

The Numerical Evaluation of a Spline from its B-Spline Representation by M. G. Cox. J. Inst. Maths. Applics (1978) 21, 135-143.

Returns

The enum `BsplineErr`.

Possible values for a `BsplineErr` are as follows:

<code>Bspline_Success</code>	Success
<code>Bspline_OutOfRange</code>	One of the arguments is out of range
<code>Bspline_OutOfRangeT</code>	The theta value is not within the limits of the t_knots
<code>Bspline_OutOfRangeZ</code>	The z value is outside the limits of the z_knots
<code>Bspline_WrongMethod1</code>	The method request must be one of : a, A, b or B
<code>Bspline_WrongMethod2</code>	The method request must be one of : a, A, b or B

4.2 BsplineAlgo_b

`BsplineAlgo_b`: The evaluation of $s(x)$ from its normalized B-spline representation using Scheme B (computation of the basis).

Synopsis

```
#include <BsplineEval/bsplineeval.h>
#include <BsplineEval/spline_coef.h>

BsplineErr BsplineAlgo_b(
    double x,
    size_t j_index,
    size_t order,
    double *c,
    double *v,
    double *knot,
    double *result
);
```

Parameters

`double x` The point to evaluate the B-spline at

`size_t j_index`
`knot[j_index-1] <= x < knot[j_index], j`

```

size_t order
    the order of the B-spline, n

double *c  the B-spline coefficients, subscripted c

double *v  work space of dimension : number order + 1

double *knot
    the B-spline knots, subscripted x

double *result
    s(x)

```

Description

The Numerical Evaluation of a Spline from its B-Spline Representation by M. G. Cox. J. Inst. Maths. Applics (1978) 21, 135-143.

Returns

The enum BsplineErr.

Possible values for a BsplineErr are as follows:

```

Bspline_Success
    Success

Bspline_OutOfRange
    One of the arguments is out of range

Bspline_OutOfRangeT
    The theta value is not within the limits of the t_knots

Bspline_OutOfRangeZ
    The z value is outside the limits of the z_knots

Bspline_WrongMethod1
    The method request must be one of : a, A, b or B

Bspline_WrongMethod2
    The method request must be one of : a, A, b or B

```

4.3 BsplineAlgo_c

BsplineAlgo_c: The evaluation of $s(t,z)$ from its normalized B-spline representation. This function allows the user to select the underlying algorithm used, 'a' or 'b'.

Synopsis

```

#include <BsplineEval/bsplineval.h>
#include <BsplineEval/spline_coef.h>

BsplineErr BsplineAlgo_c(

```

```

    double theta,
    double z,
    BsplineInput *bspline_input,
    double *result
);

```

Parameters

```

double theta
    The value of theta to evaluate the B-spline

double z
    The value of z to evaluate the B-spline

BsplineInput *bspline_input
    All necessary B-spline input

double *result
    s( theta, z )

```

Description

The Numerical Evaluation of a Spline from its B-Spline Representation by M. G. Cox. J. Inst. Maths. Applics (1978) 21, 135-143.

Returns

The enum BsplineErr. SEE ALSO Bspline_eval

Possible values for a BsplineErr are as follows:

```

Bspline_Success
    Success

Bspline_OutOfRange
    One of the arguments is out of range

Bspline_OutOfRangeT
    The theta value is not within the limits of the t_knots

Bspline_OutOfRangeZ
    The z value is outside the limits of the z_knots

Bspline_WrongMethod1
    The method request must be one of : a, A, b or B

Bspline_WrongMethod2
    The method request must be one of : a, A, b or B

```

4.4 BsplineAlgo_c_derivs

BsplineAlgo_c_derivs – The evaluation of $s(t,z)$ and the derivative with respect to t and z from its normalized B-spline representation. This function allows the user to select the underlying algorithm used, 'a' or 'b'.

Synopsis

```
#include <BsplineEval/bsplineeval.h>
#include <BsplineEval/spline_coef.h>

BsplineErr BsplineAlgo_c_derivs(
    double theta,
    double z,
    BsplineInput *bspline_input,
    BsplineResult *bspline_result
);
```

Parameters

```
double theta
    The theta to evaluate the B-spline

double z
    The value of z to evaluate the B-spline

BsplineInput *bspline_input
    All B-spline input

BsplineResult *bspline_result
    All B-spline output
```

Description

The Numerical Evaluation of a Spline from its B-Spline Representation by M. G. Cox. J. Inst. Maths. Applics (1978) 21, 135-143.

Returns

The enum BsplineErr.

Possible values for a BsplineErr are as follows:

```
Bspline_Success
    Success

Bspline_OutOfRange
    One of the arguments is out of range

Bspline_OutOfRangeT
    The theta value is not within the limits of the t_knots

Bspline_OutOfRangeZ
    The z value is outside the limits of the z_knots

Bspline_WrongMethod1
    The method request must be one of : a, A, b or B

Bspline_WrongMethod2
    The method request must be one of : a, A, b or B
```

4.5 Bspline_determine_index

1] <= key < base[index]. If key == base[num - 1] then index is set to num - 1.

Synopsis

```
#include <BsplineEval/bsplineeval.h>
#include <BsplineEval/spline_coef.h>

BsplineErr Bspline_determine_index(
    double *key,
    void *base,
    size_t num,
    double minval,
    double maxval,
    size_t *index
);
```

Parameters

```
double *key
    The key to search for

void *base
    Points to the element at the base of array

size_t num
    Number of elements in the array

double minval
    The left most knot boundary

double maxval
    The right most knot boundary

size_t *index
    base[ index - 1 ] <= key < base[ index ]
```

Description

Do a Binary search to find the location of the key within the knots. If this routine is usually called iteratively between the ideal surface and the deformation map then it can improve a bit by making use of the last position found.

Returns

The enum BsplineErr.

Possible values for a BsplineErr are as follows:

<code>Bspline_Success</code>	Success
<code>Bspline_OutOfRange</code>	One of the arguments is out of range
<code>Bspline_OutOfRangeT</code>	The theta value is not within the limits of the <code>t_knots</code>
<code>Bspline_OutOfRangeZ</code>	The z value is outside the limits of the <code>z_knots</code>
<code>Bspline_WrongMethod1</code>	The method request must be one of : a, A, b or B
<code>Bspline_WrongMethod2</code>	The method request must be one of : a, A, b or B

4.6 Bspline_err_msg

`Bspline_err_msg` – print the error message to the file pointer.

Synopsis

```
#include <BsplineEval/bsplineeval.h>
#include <BsplineEval/spline_coef.h>

void Bspline_err_msg(
    BsplineErr err,
    FILE *fp
);
```

Parameters

`BsplineErr err`

Possible values for a `BsplineErr` are as follows:

<code>Bspline_Success</code>	Success
<code>Bspline_OutOfRange</code>	One of the arguments is out of range
<code>Bspline_OutOfRangeT</code>	The theta value is not within the limits of the <code>t_knots</code>
<code>Bspline_OutOfRangeZ</code>	The z value is outside the limits of the <code>z_knots</code>
<code>Bspline_WrongMethod1</code>	The method request must be one of : a, A, b or B

`Bspline_WrongMethod2`

The method request must be one of : a, A,
b or B

FILE *fp Not Documented.

Description

`Bspline_err_msg` – print the error message to the file pointer.

Returns

void.

4.7 Bspline_eval

`Bspline_eval`: The evaluation of $s(t,z)$ from its normalized B-spline representation. This function is similar to the function `BsplineAlgo_c` with the exception that the underlying algorithms 'b' and 'b' are chosen for the user.

Synopsis

```
#include <BsplineEval/bsplineeval.h>
#include <BsplineEval/spline_coef.h>

BsplineErr Bspline_eval(
    double theta,
    double z,
    BsplineInput *bspline_input,
    double *result
);
```

Parameters

```
double theta
    The value of theta to evaluate the B-spline

double z
    The value of z to evaluate the B-spline

BsplineInput *bspline_input
    All necessary B-spline input

double *result
    s( theta, z )
```

Description

The Numerical Evaluation of a Spline from its B-Spline Representation by M. G. Cox. J. Inst. Maths. Applics (1978) 21, 135-143.

Returns

The enum `BsplineErr`.

Possible values for a `BsplineErr` are as follows:

<code>Bspline_Success</code>	Success
<code>Bspline_OutOfRange</code>	One of the arguments is out of range
<code>Bspline_OutOfRangeT</code>	The theta value is not within the limits of the t_knots
<code>Bspline_OutOfRangeZ</code>	The z value is outside the limits of the z_knots
<code>Bspline_WrongMethod1</code>	The method request must be one of : a, A, b or B
<code>Bspline_WrongMethod2</code>	The method request must be one of : a, A, b or B

4.8 Bspline_eval_derivs

`Bspline_eval_derivs` – The evaluation of $s(t,z)$ and the derivative with respect to t and z from its normalized B-spline representation. This function is an optimized version of the function `BsplineAlgo_c_derivs`.

Synopsis

```
#include <BsplineEval/bsplineval.h>
#include <BsplineEval/spline_coef.h>

BsplineErr Bspline_eval_derivs(
    double theta,
    double z,
    BsplineInput *bspline_input,
    BsplineResult *bspline_result
);
```

Parameters

```
double theta    The theta to evaluate the B-spline

double z       The value of z to evaluate the B-spline

BsplineInput *bspline_input
               All B-spline input
```

```

BsplineResult *bspline_result
    All B-spline output

```

Description

An optimized implementation of the function `BsplineAlgo_c`. This function can be further optimized by folding in the last of the separate call to `BsplineAlgo_b`, but this requires the rewriting of the `algo_b` routine. The Numerical Evaluation of a Spline from its B-Spline Representation by M. G. Cox. *J. Inst. Maths. Applics* (1978) 21, 135-143.

Returns

The enum `BsplineErr`.

Possible values for a `BsplineErr` are as follows:

```

Bspline_Success
    Success

Bspline_OutOfRange
    One of the arguments is out of range

Bspline_OutOfRangeT
    The theta value is not within the limits of the t_knots

Bspline_OutOfRangeZ
    The z value is outside the limits of the z_knots

Bspline_WrongMethod1
    The method request must be one of : a, A, b or B

Bspline_WrongMethod2
    The method request must be one of : a, A, b or B

```

4.9 Bspline_free

`Bspline_free` – free memory allocated by `Bspline_init`.

Synopsis

```

#include <BsplineEval/bsplineeval.h>
#include <BsplineEval/spline_coef.h>

void Bspline_free(BsplineInput *input);

```

Parameters

```

BsplineInput *input
    Not Documented.

```

Description

Bspline_free – free memory allocated by Bspline_init.

Returns

void.

4.10 Bspline_init

Allocates the memory used for Bspline library.

Synopsis

```
#include <BsplineEval/bsplineval.h>
#include <BsplineEval/spline_coef.h>

BsplineInput *Bspline_init(spline_coef *scoef);
```

Parameters

```
spline_coef *scoef
    Not Documented.
```

Description

Allocates the memory used for Bspline library.

Returns

BsplineInput* if successful else NULL is returned.

5 A Timing Comparison of BsplineEval library vs DASL

The file ./test/p6.spl with the following characteristics:

```
# spline deformation amplitudes
# z_order      4
# theta_order  4
# qz           421
# num_z_knots  417
# num_theta_knots 144
# zmin zmax    -1.000000e+00 1.000000e+00
# tmin tmax    0.000000e+00 6.283185e+00
# rms amplitude 129188.64688685712463
#
```

will be used to derive the following timing results for evaluating the spline without evaluating the derivatives and then with derivatives. The actual time one may get is dependent on many factors such as the number of theta and z knots etc...

	6	6	6
	10	4 * 10	9 * 10

ab	35.7	1:58.9	4:17.5
ba	33.6	1:50.7	3:59.0
aa	36.2	2:01.4	4:23.2
bb	32.9	1:48.2	3:53.3
optimized	33.2	1:48.9	3:55.5
dasl	59.8	3:35.5	7:54.3

Note that the time for the optimized version (without evaluating the derivatives) is slightly more than the time for the 'bb' version. The reason for this is the routine `Bspline_eval` was provided as a symmetry to the function `Bspline_eval_derivs`. The routine `Bspline_eval` sets to flag, `method1='b'`; and `method2='b'`; before calling routine 'bb' so a slightly slower time is expected.

	6	6	6
	10	4 * 10	9 * 10

ab	1:09.8	4:15.6	9:24.8
ba	1:04.5	3:54.5	8:37.9
aa	1:11.4	4:21.9	9:39.4
bb	1:02.8	3:47.7	8:22.7

optimized		45.5		2:38.1		5:45.6
dasl		2:38.2		10:09.0		22:40.0

The script `./test/timeit.ksh` was used to generate the entries to this table.

The time gain in the `dasl` routine versus the optimized Algorithm C is dependent on several factors. The order of the spline and then number of knots in `theta` and `z`.

6 Testing the BsplineEval library vs DASL

Testing was performed by comparing the results given by the dasl and the optimized algorithm C. The paragraphs to follow show the results of two different input files, one with ‘large’ rms deformation (p6.spl) and one with ‘small’ rms deformation (p6_low_31_72.SPL). The results agree to within roundoff errors, this was done by running the following command:

The file ./test/p6.spl with the following characteristics:

```
# spline deformation amplitudes
# z_order          4
# theta_order     4
# qz              421
# num_z_knots     417
# num_theta_knots 144
# zmin zmax      -1.000000e+00 1.000000e+00
# tmin tmax      0.000000e+00 6.283185e+00
# rms amplitude   129188.64688685712463
#
```

yield the following results:

```
futile-90: objs/sun4u-SunOS-5/evalbspline num=100 test=5 \
| column -v -a deltas N - | column -v -a deltadt N \
| column -v -a deltadz N \
    | compute deltas = \( dasl - algoc \) \ / dasl \
    | compute deltadz = \( ddasldz - dalgocdz \) \ / ddasldz \
    | compute deltadt = \( ddasldt - dalgocdt \) \ / ddasldt \
    | rdbstats deltas deltadz deltadt | tbl2lst

    deltas_n | N
deltas_sum | N
deltas_ave | N
deltas_dev | N
deltas_min | N
deltas_max | N
deltadz_n | N
deltadz_sum | N
deltadz_ave | N
deltadz_dev | N
deltadz_min | N
deltadz_max | N
deltadt_n | N
deltadt_sum | N
deltadt_ave | N
deltadt_dev | N
```

```

deltadt_min | N
deltadt_max | N

      deltas_n | 10000
deltas_sum  | -1.29628283548978e-13
deltas_ave  | -1.29628283548978e-17
deltas_dev  | 1.10245039134638e-16
deltas_min  | -5.83761910253855e-16
deltas_max  | 5.3919301269975e-16
      deltadz_n | 10000
deltadz_sum | -9.28799965664604e-14
deltadz_ave | -9.28799965664602e-18
deltadz_dev | 1.33699222283282e-16
deltadz_min | -3.90195209692813e-15
deltadz_max | 3.74917737998133e-15
      deltadt_n | 10000
deltadt_sum | -1.46179231627943e-13
deltadt_ave | -1.46179231627943e-17
deltadt_dev | 1.07162272062163e-16
deltadt_min | -6.82814047720486e-16
deltadt_max | 7.42207044201643e-16

```

The file ./test/p6_low_31_72.SPL with the following characteristics:

```

# spline deformation amplitudes
# z_order      4
# theta_order  4
# qz           33
# num_z_knots  29
# num_theta_knots 71
# zmin zmax    -1.000000e+00 1.000000e+00
# tmin tmax    0.000000e+00 6.283185e+00
# rms amplitude 0.00101284258284

```

yield the following results:

```

futile-93: objs/sun4u-SunOS-5/evalbspline num=100 test=5 \
input=p6_low_31_72.SPL \
| column -v -a deltas N - | column -v -a deltadt N \
| column -v -a deltadz N \
      | compute deltas = \( dasl - algoc \) \ / dasl \
      | compute deltadz = \( ddasldz - dalgocdz \) \ / ddasldz \
      | compute deltadt = \( ddasldt - dalgocdt \) \ / ddasldt \
      | rdbstats deltas deltadz deltadt | tbl2lst

```

```
    deltas_n | N
deltas_sum | N
deltas_ave | N
deltas_dev | N
deltas_min | N
deltas_max | N
    deltadz_n | N
deltadz_sum | N
deltadz_ave | N
deltadz_dev | N
deltadz_min | N
deltadz_max | N
    deltadt_n | N
deltadt_sum | N
deltadt_ave | N
deltadt_dev | N
deltadt_min | N
deltadt_max | N

    deltas_n | 10000
deltas_sum | -0
deltas_ave | 0
deltas_dev | 0
deltas_min | -0
deltas_max | -0
    deltadz_n | 10000
deltadz_sum | 0
deltadz_ave | 0
deltadz_dev | 0
deltadz_min | 0
deltadz_max | 0
    deltadt_n | 10000
deltadt_sum | 0
deltadt_ave | 0
deltadt_dev | 0
deltadt_min | -0
deltadt_max | -0
```


7 Known Problems and Improvements to be made

In evaluating $s(\theta, z)$ from its normalized representation, the library currently assumes θ to be periodic. The library can ‘easily’ be made to be more general, ie the library can should allow the user to specify the which variable is periodic or none at all.

The routine `eval_Bspline_derivs` is an optimized version of the routine `BsplineAlgo.c`. This was done by folding one of the calls to calculate the derivative into the call to evaluate $s(\theta, z)$. It will require a little work to fold in the other derivative. A 10-15% savings is expected.

The library currently needs three arrays of B-spline coefficients (`scoef->c`, `scoef->cdifz`, `scoef->cdift`), a more optimized data structure would be to pass the library an array of structures. Theoretically, this will minimize caching in results since only n number (where n is the order of the B-spline) of elements are needed to evaluate the spline.

